# Spacetime Defects: von Kármán vortex street like configurations

Patricio S. Letelier[1]

Departamento de Matemática Aplicada-IMECC
Universidade Estadual de Campinas
13081-970 Campinas. S.P., Brazil



**Abstract**

A special arrangement of spinning strings with dislocations similar to a von Kármán vortex street is studied. We numerically solve the geodesic equations for the special case of a test particle moving along two infinite rows of pure dislocations and also discuss the case of pure spinning defects.

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Conical singularities or spacetime defects are characterized by Riemann-Christoffel curvature tensor, or Cartan torsion, or both different from zero only on the subspace (event, world line, world sheet, or world tube) that describes the evolution of the defect (texture, monopole, string, or membrane). In other words we have that the curvature, the torsion, or both, are proportional to distributions with support on the defect. Spacetimes with conical singularities of different types has been studied recently in a variety of contexts, e.g. spinning strings with cosmic dislocations [1][2], pure spacetime dislocations [3]– [5], also in low dimensional gravity [6]. For the discussion of a great variety of defects see Ref. [7].

Some quantum aspects related to line defects has been considered by several authors: The spectra of a quantum particle in the presence of

[1]e-mail: letelier@ime.unicamp.br

conical singularities that physically corresponds to a cosmic strings, a screw dislocation, and a rotating string [8]. The harmonic interaction of a particle in the presence of defects [9]. A Berry quantum phase for particles transported along a defect [10]. The relation with the chiral anomaly is Ashtekar approach to quantum gravity [11]. Classical aspects has been also recently considered: The gravitational energy of the conical defects [12]. Geodesic motions around line defects [13]. The magnetic and electric self-forces on a straight wire induced by a topological line defect carrying singular torsion [14].

The purpose of this note is to study a particular infinite superposition of line defects that can be described as a von Kármán vortex street [17] kind of arrangement of spinning cosmic strings with dislocations. This arrangement of strings is depicted in Fig. 1. We have two rows of an infinite number of strings, the coordinate distance between the rows is $2b$ and the strings in a row are apart by a distance $2a$. One row is displaced in an amount $a$ with respect to the other. All the strings in the same row have equal spins and equal dislocations (the spins and the dislocations are not necessarily equal). The strings in the two rows have opposite spins and dislocations.

First, let us summarize the main relations associated to a spinning string with dislocation. The associated spacetime is [2],

$$ds^2 = (\omega^t)^2 - (\omega^z)^2 - (\omega^x)^2 - (\omega^y)^2, \tag{1}$$

with

$$\omega^t = dt - \partial_y W dx + \partial_x W dy, \tag{2}$$

$$\omega^z = dz - \partial_y U dx + \partial_x U dy, \tag{3}$$

$$\omega^x = e^{-2V} dx, \;\; \omega^y = e^{-2V} dy, \tag{4}$$

where the the functions: $U, W$ and $V$ are

$$W = 4\sigma f_W, \quad U = 4\kappa f_U, \quad V = 2\lambda f_V, \tag{5}$$

and for the case of a single defect all the three functions, $f_W,\ f_U$ and $f_V$ are equal

$$f = \ln r, \quad (r \equiv +\sqrt{x^2 + y^2}). \tag{6}$$

In the context of the Riemann-Cartan geometry[15] this spacetime has curvature and torsion that are proportional to Dirac distributions with support on the line $x = y = 0$. We have that the tetradical components of the curvature tensor and the torsion reduce to:

$$R_{xyxy} = 2e^{4V}(\partial_{xx} + \partial_{yy})V = 8\pi\lambda\delta(x)\delta(y)/\sqrt{-g}, \tag{7}$$

$$S_{txy} = (\partial_{xx} + \partial_{yy})W = 8\pi\sigma\delta(x)\delta(y), \tag{8}$$

$$S_{zxy} = (\partial_{xx} + \partial_{yy})U = 8\pi\kappa\delta(x)\delta(y). \tag{9}$$

¿From Einstein-Cartan equations one finds that this metric represents a single string locate along the $z$-axis with linear density, $\lambda$ [2]. The string has as equation of state: linear density equal to tension, i.e., the equation of state for usual cosmic strings. The string is spinning with an "angular velocity" $\sigma$ and has a dislocation given by $\kappa$. In this case, the analogue of the Burgers vector of dislocation has a single component along the $z$-axis, $2\kappa/\pi$.

Note that we have three independent structures: i) A disclination or cosmic string, ii) A "time dislocations" or spin, and ii) A space dislocation or cosmic dislocation. Each one of these structures gives us a singularity along the $z$-axis (line singularity). Thus, when these three structures are one in top of the other we have the a spinning string with cosmic dislocation. But we can also have: a) A string with dislocation only, $\sigma = 0$, b) A spinning string,

$\kappa = 0$, c) A spinning cosmic dislocation, $\lambda = 0$, d) A pure time dislocation or spinning defect, $\lambda = \kappa = 0$, e) A pure cosmic dislocation, $\lambda = \sigma = 0$, and f) A usual cosmic string, $\kappa = \sigma = 0$. The gravitational energy of these defects were considered in [12] and the spacetime analogue of other Volterra distortions in [16].

Now we shall consider single row of strings form by identical strings parallel to the $z$-axis that cross the $x$-axis, at: $\cdots - 4a, -2a, 0, +2a, +4a, \cdots$. The function $f_V$ in this case is

$$f_V = \frac{1}{2} \sum_{n=-\infty}^{\infty} \ln[(x - 2na)^2 + y^2]. \tag{10}$$

This function gives right curvature, since

$$(\partial_{xx} + \partial_{yy}) f_V = 2\pi \sum_{n=-\infty}^{\infty} \delta(x - 2na)\delta(y). \tag{11}$$

Introducing the complex variable, $\zeta = x + iy$, we have,

$$\begin{aligned} 2f_V &= \sum_{n=-\infty}^{\infty} \ln[(\zeta - 2an)(\bar{\zeta} - 2an)], && (12)\\ &= \ln(\zeta\bar{\zeta}) + \sum_{n=1}^{\infty} \ln[(\zeta^2 - 4a^2n^2)(\bar{\zeta}^2 - 4a^2n^2)], && (13)\\ &= 2\widehat{f}_V + 4\sum_{n=1}^{\infty} \ln(2an) && (14) \end{aligned}$$

with

$$2\widehat{f}_V = \ln\{\zeta \prod_{n=1}^{\infty} [1 - (\zeta/2an)^2] \bar{\zeta} \prod_{m=1}^{\infty} [1 - (\bar{\zeta}/2am)^2]\} \tag{15}$$

Since the singular structure of (11) does not change if we add a constant to $f_V$ we have that (14) and (15) represent the same type of defects. So we shall use this last function to represent the infinite row of strings. By using

the identity [18],

$$\sin x = x \prod_{n=1}^{\infty} [1 - (x/n\pi)^2], \tag{16}$$

$\widehat{f}_V$ can be cast in the simple form,

$$\begin{aligned} \widehat{f}_V(x,y) &= \ln|\sin[\pi(x+iy)/2a]|, & (17)\\ &= \frac{1}{2}\ln[\cosh^2(\pi y/2a) - \cos^2(\pi x/2a)]. & (18) \end{aligned}$$

Thus the functions $V = \lambda \widehat{f}_V(x,y)$ and $W = U = 0$ are the metric potentials that describe an infinite row of cosmic strings. With $\widehat{f}_V(x, y-b)$ we have the upper row of Fig. 1 with no spin and dislocation. To put spin and dislocation to each string we do $W = \sigma \widehat{f}_V(x, y-b)$ and $U = \kappa\sigma \widehat{f}_V(x, y-b)$. The other row is represented by the functions $V = \lambda \widehat{f}_V(x+a, y+b)$, $W = -\sigma \widehat{f}_V(x+a, y+b)$ and $U = -\kappa \widehat{f}_V(x+a, y+b)$ (the spins, as well as, the dislocations of the defects of the two rows are opposite). Now the metric potentials for the whole von Kármán vortex street type of distribution presented in Fig. 1 are: $W = \sigma[\widehat{f}_V(x, y-b) - \widehat{f}_V(x+a, y+b)]$ , $U = \kappa[\widehat{f}_V(x, y-b) - \widehat{f}_V(x+a, y+b)]$ , and $V = \lambda[\widehat{f}_V(x, y-b) + \widehat{f}_V(x+a, y+b)]$ , i.e.,

$$W = 2\sigma \ln \frac{\cosh^2[\pi(y-b)/2a] - \cos^2(\pi x/2a)}{\cosh^2[\pi(y+b)/2a] - \sin^2(\pi x/2a)}, \tag{19}$$

$$U = 2\kappa \ln \frac{\cosh^2[\pi(y-b)/2a] - \cos^2(\pi x/2a)}{\cosh^2[\pi(y+b)/2a] - \sin^2(\pi x/2a)}, \tag{20}$$

$$\begin{aligned} V = \lambda \ln\{&(\cosh^2[\pi(y-b)/2a] - \cos^2(\pi x/2a)) \times \\ &(\cosh^2[\pi(y+b)/2a] - \sin^2(\pi x/2a))\} \end{aligned} \tag{21}$$

In order to better understand this metric we shall study the motion of test particles traveling between the two rows of strings. Since the torsion is a distribution with support only on the defect we have that, in this case,

geodesics and autoparallels are the same. The geodesic equations for the metric (1)-(4) reduce to:

$$\dot{t} - \dot{x}\partial_y W + \dot{y}\partial_x W = K_1, \tag{22}$$

$$\dot{z} - \dot{x}\partial_y U + \dot{y}\partial_x U = K_2, \tag{23}$$

$$\ddot{x} - 2(\dot{x}^2 - \dot{y}^2)\partial_x V - 4\dot{x}\dot{y}\partial_y V = 0, \tag{24}$$

$$\ddot{y} - 2(\dot{x}^2 - \dot{y}^2)\partial_y V - 4\dot{x}\dot{y}\partial_x V = 0, \tag{25}$$

where $K_1$ and $K_2$ are constants of integration. From the metric (1)-(4) we obtain the relation,

$$(\dot{x}^2 + \dot{y}^2)e^{-4V} = -1 + K_1^2 - K_2^2. \tag{26}$$

We note that the motion in the plane $(x, y)$ is independent of the variable $z$ and is determined mainly by the string part. The motion in this plane determines the motion along $z$. This last motion is due solely to the dislocation. Since this part will represent the major depart from the usual cosmic strings we shall first analyze the case: $\lambda = \sigma = 0$, i.e., an arrangement of pure dislocations. Now the geodesic equations are: (23), $\dot{t} = K_1$, and $\ddot{x} = \ddot{y} = 0$. The last two equations tell us that the motion is confined to a plane parallel to the $z$-axis. We integrate these equation and choose the constant of integrations in such a way that the particle be confined between the two rows of dislocations: $y = y_0$ with $-b < y_0 < b$ and $x = v_x s + x_0$. The constraint (26) now reads,

$$K_1^2 = 1 + K_2^2 + v_x^2. \tag{27}$$

Thus, it remains only one equation to solve, Eq. (23), that we shall study numerically. In Fig. 2 we present the geodesic motion of a test particle in the gravitational field of a von Kármán street type of configuration constructed

with cosmic dislocations for different values of the constants: $y_0 = -1.8$ (lower curve near the origin), $y_0 = 0$ (middle curve) and $y_0 = 1.8$ (upper curve); $x_0 = 0$ for the three cases. The value of the parameters are $a = 1$, $b = 2$, $\kappa = -0.45$. We take for the integration constants: $v_x = 0.8$ and $K_2 = 0$ [note that $K_1$ is determined by Eq. (27)]. The three curves intersect in points that have the same coordinate $x$ as the dislocations.

In summary, with no cosmic dislocations we have a motion on the plane $z = 0$ parallel to the $x$-axis. The presence of the pure cosmic dislocations makes the particle leave this plane and acquire velocity along the $z$ direction. Depending on the sign, the dislocations may accelerate or brake the motion of the particle in the $z$-direction. We have a very symmetric arrangement of dislocations that is reflected in the motion of the particles.

Finally, let us turn our attention to the case of pure spinning defects, in this case we have $\kappa = \lambda = 0$. Then the particles move always in a strait line. If we take $\sigma = -0.45$ and the same values for the rest of the constants as in the precedent case, we find that the test particles do not leave the plane $(x, y)$ and move parallel to the $x$-axis. The time coordinate is “accelerated” and “braked” as the $z$-coordinate we have the same behaviour shown in Fig. 2 with the $z$-axis changed by the coordinate $t$. Again the symmetry of the arrangement of pure spinning defects is reflected in the motion of the test particles.

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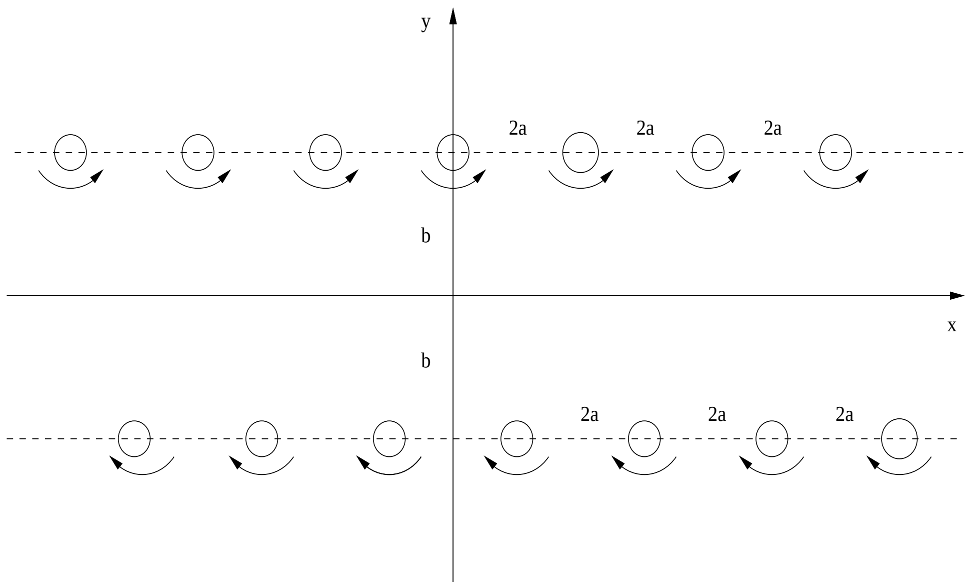


Figure 1: A von Kármán vortex street like distribution of spinning cosmic strings with dislocations is depicted. All the strings in the same row have equal spins and equal dislocations (the spins and the dislocations are not necessarily equal). The strings in the two rows have opposite spins and dislocations.

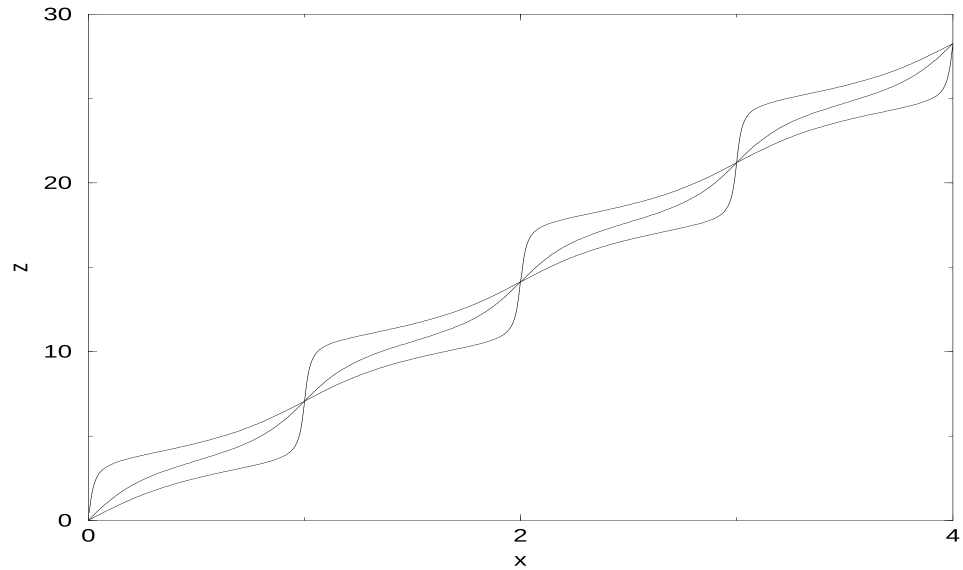


Figure 2: Geodesic motions of a test particle in a von Kármán street type of distribution of cosmic dislocations for different values of the constants: $y_0 = -1.8$ (lower curve near the origin), $y_0 = 0$ (middle curve) and $y_0 = 1.8$ (upper curve); $x_0 = 0$ for the three cases. The value of the parameters are $a = 1,\ b = 2,\ \kappa = -0.45$. We take for the integration constants: $v_x = 0.8$ and $K_2 = 0$.